

# RESTRICTED VOLUMES AND DIVISORIAL ZARISKI DECOMPOSITIONS

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**ABSTRACT.** We give a relation between the existence of a Zariski decomposition and the behavior of the restricted volumes of a big divisor on a smooth complex projective variety. Moreover we give the formula expressing a restricted volume with current integration by using analytic methods. This implies that we can define the restricted volume of a transcendental class on a compact Kähler manifold in natural way. The similar relation can be extended to a transcendental class.

## 1. INTRODUCTION

In this paper,  $X$  denotes a smooth complex projective variety of dimension  $n$ ,  $D$  a (big) divisor on  $X$  and  $V \subseteq X$  an irreducible subvariety of dimension  $d$ . Then the restricted volume of  $D$  along  $V$  is defined to be

$$\mathrm{vol}_{X|V}(D) = \limsup_{k \rightarrow \infty} \frac{\dim \mathrm{Im} \left( H^0(X, \mathcal{O}_X(kD)) \longrightarrow H^0(V, \mathcal{O}_V(kD)) \right)}{k^d/d!}.$$

Roughly speaking restricted volumes measure the number of sections of  $\mathcal{O}_V(kD)$  which can be extended to  $X$ . Restricted volumes have many applications in various situations. They are studied in [ELMNP], [BFJ09] and so on.

On the other hand, it is important problem when  $D$  admits a Zariski decomposition. Here  $D$  is said to admit a Zariski decomposition if there exist a nef  $\mathbb{R}$ -divisor  $P$  and an effective  $\mathbb{R}$ -divisor  $N$  such that following map is an isomorphism for every integer  $k > 0$ :

$$H^0(X, \mathcal{O}_X(\lfloor kP \rfloor)) \longrightarrow H^0(X, \mathcal{O}_X(kD)).$$

The map is multiplying the section  $e_k$ , where  $e_k$  is the standard section of the effective divisor  $\lceil kN \rceil$ . Here  $\lfloor G \rfloor$  (*resp.*  $\lceil G \rceil$ ) denotes round down (*resp.* round up) of an  $\mathbb{R}$ -divisor  $G$ .

Assume  $V$  is not contained in  $\mathbb{B}_+(D)$ . Here  $\mathbb{B}_+(D)$  denotes the augmented base locus of  $D$ . It is the closed analytic set in  $X$ , which is the defect of  $D$  to be ample. Then the restricted volume  $\mathrm{vol}_{X|V}(\cdot)$  along  $V$  depends only on the first Chern class (the numerical class) of  $D$ . One ask whether the restricted volume  $\mathrm{vol}_{X|V} D$  of  $D$  depends only on the numerical class of  $V$ . [BFJ09] shows the question is affirmative

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when the codimension of  $V$  is one. In this paper, we give a necessary and sufficient condition for  $D$ , that the restricted volume of  $D$  depends only on the numerical class of  $V$ . The condition is related to the existence of a Zariski decomposition of  $D$  as follows:

**Theorem 1.1.** *Assume that  $D$  is big. The following two conditions are equivalent.*

- (1)  *$D$  admits a Zariski decomposition.*
- (2)  *$\text{vol}_{X|V}(D) = \text{vol}_{X|V'}(D)$  holds for any pair of subvarieties  $V$  and  $V'$  of  $X$  such that  $V \sim V'$  and  $V, V' \not\subseteq \mathbb{B}_+(D)$ .*

When  $V$  and  $V'$  are numerically equivalent, we write  $V \sim V'$ . Roughly speaking, the condition (2) claims that the restricted volume  $\text{vol}_{X|V}(D)$  of  $D$  is determined only by the numerical class of  $V$ . Theorem 1.1 implies that the restricted volumes along some numerically equivalent subvarieties  $V$  and  $V'$  can be different when  $D$  does not admit a Zariski decomposition. Such example is given by so-called Cutkosky construction.

When  $V$  is the ambient space  $X$ , the restricted volume of  $D$  is the usual volume  $\text{vol}_X(D)$  of  $D$ . The usual volume has been studied by several authors. The general theory is presented in detail and with full references in [La04]. Boucksom expressed the usual volume  $\text{vol}_X(D)$  by current integration, using the Calabi-Yau technique to solve Monge-Ampère equations and the singular holomorphic Morse inequalities (see [Bou02]). In other words, Boucksom expressed the volume of  $D$  in terms of only the first Chern class  $c_1(D)$ .

Restricted volumes along  $V$  depend only on the first Chern class  $c_1(D)$  if  $V$  is not contained in the augmented base locus  $\mathbb{B}_+(D)$  of  $D$ . Then Boucksom's formula can be extended to restricted volumes as follows:

**Theorem 1.2.** *Assume that  $V$  is not contained in the augmented base locus  $\mathbb{B}_+(D)$ . Then the restricted volume of  $D$  along  $V$  satisfies the following equality:*

$$\text{vol}_{X|V}(D) = \sup_{T \in c_1(D)} \int_{V_{\text{reg}}} (T|_{V_{\text{reg}}})_{\text{ac}}^d,$$

for  $T$  ranging among a positive  $(1,1)$ -current with analytic singularities whose singular locus does not contain  $V$ .

Here  $T|_{V_{\text{reg}}}$  denotes the restriction of  $T$  to the regular locus  $V_{\text{reg}}$  of  $V$  and  $(T|_{V_{\text{reg}}})_{\text{ac}}$  denotes the absolutely continuous part (see subsection 2.4). Moreover we can express the restricted volumes with non-pluripolar product, which is introduced in [BEGZ]. Theorem 1.2 implies that we can define the restricted volume of a transcendental class on a compact Kähler manifold in natural way.

**Definition 1.3.** Let  $V$  be an irreducible variety of dimension  $d$  on a compact Kähler manifold  $M$  and  $\alpha$  the class in  $H^{1,1}(M, \mathbb{R})$ . Assume that  $V$  is not contained in the non-Kähler locus  $E_{nK}(\alpha)$ . Then the restricted volume of  $\alpha$  along  $V$  is defined to be

$$\text{vol}_{M|V}(\alpha) = \sup_{T \in \alpha} \int_{V_{\text{reg}}} (T|_{V_{\text{reg}}})_{\text{ac}}^d,$$

for  $T$  ranging among a positive  $(1, 1)$ -current with analytic singularities and the singular locus of  $T$  does not contain  $V$ .

Here the non-Kähler locus is analytic counter part of the augmented base locus. When  $\alpha$  is the Chern class of some divisor  $D$ , the non-Kähler locus  $E_{nK}(\alpha)$  coincides with the augmented base locus  $\mathbb{B}_+(D)$ . For this extended definition, the properties of usual restricted volumes hold. For example, the continuity, log concavity, Fujita's approximations and so on (see subsection 4.2). Moreover the analogue of Theorem 1.1 holds for this extended definition as follows. The principle of the proof is essentially same as Theorem 1.1. The proof gives another proof of Theorem 1.1 by using analytic methods (see subsection 4.3).

**Theorem 1.4.** *Let  $\alpha$  be a big class in  $H^{1,1}(X, \mathbb{R})$ . Then the following two conditions are equivalent.*

- (1)  $\alpha$  admits a Zariski decomposition.
- (2)  $\text{vol}_{X|V}(\alpha) = \text{vol}_{X|V'}(\alpha)$  holds for any pair of subvarieties  $V$  and  $V'$  of  $X$  such that  $V, V'$  defines the same cohomology class and  $V, V' \not\subseteq E_{nK}(\alpha)$ .

Here the Zariski decomposition of a class  $\alpha$  is defined by divisorial Zariski decompositions introduced in [Bou04] (see subsection 2.7). When  $\alpha$  is the first Chern class of some divisor, the Zariski decomposition of  $\alpha$  correspond with that of the divisor.

*Remark 1.5.* The class  $\alpha$  is not always contained in the Néron-Severi group. Therefore Theorem 1.4 is essentially stronger statement than Theorem 1.1.

In the proof of Theorem 1.1 and 1.4, to consider the restricted base locus of a divisor is important. The restricted base locus  $\mathbb{B}_-(D)$  of  $D$  is a countable union of closed analytic set, which is the defect of  $D$  to be nef. In section 5, we accurize the simplest version of Kawamata-Viehweg vanishing theorem in response to the dimension of the restricted base locus. This proof is essentially to use Kawamata-Viehweg vanishing theorem (see [Ka82], [Vie82]). A divisor  $D$  is said to be nef in codimension  $k$  if the codimension of non-nef locus greater than  $k$ .

**Theorem 1.6.** *Assume that  $D$  is nef in codimension  $k$  and big. Then*

$$H^q(X, \mathcal{O}_X(K_X + D)) = 0 \quad \text{for any } q \geq n - k.$$

*Remark 1.7.* Note  $D$  is nef in codimension  $n - 1$  if and only if  $D$  is nef. Then Theorem 1.6 implies  $H^q(X, \mathcal{O}_X(K_X + D)) = 0$  ( $q \geq 1$ ). This is the simplest version of Kawamata-Viehweg vanishing theorem.

This theorem can be generalized by using the generalized numerical Kodaira dimension  $\nu(D)$  of  $D$ , which is introduced in [BDPP]. Theorem 1.6 implies an asymptotic estimate of higher cohomology as follows:

**Corollary 1.8.** *Let  $E$  be an arbitrary divisor on  $X$ . Assume that  $D$  is nef in codimension  $k$ . Then the following estimate holds for  $q \geq n - k$  as  $\ell \rightarrow \infty$ :*

$$\dim H^q(X, \mathcal{O}_X(E + \ell D)) = O(\ell^{n-q}).$$

This estimate is similar to [Ku06, Proposition 2.15]. Corollary 1.8 and asymptotic Riemann-Roch formula shows that the inequality  $\text{vol}_{X|X}(D) \geq D^n$  holds when  $D$  nef in codimension  $n - 2$ . In particular, then  $D$  is big when the self-intersection number  $D^n$  is positive. In the case that  $X$  is surface, this fact is well-known.

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## 2. TERMINOLOGY

**2.1. Currents.** Let  $M$  be a compact Kähler manifold in this section. Then  $H^{p,p}(M, \mathbb{C})$  is identified with the quotient of the space of  $d$ -closed  $(p, p)$ -currents modulo the  $dd^c$ -exact currents. For our purpose the case of  $p = 1$  is important. A  $d$ -closed almost positive  $(1, 1)$ -current  $T$  is said to have analytic (*resp.* algebraic) singularities (along a subscheme  $V(\mathcal{I})$  defined by an ideal sheaf  $\mathcal{I}$ ), if its potential function  $\varphi$  can be locally written as

$$\varphi = \frac{c}{2} \log(|f_1|^2 + \dots + |f_k|^2) + v$$

for some  $c \in \mathbb{R}_{>0}$  (*resp.*  $c \in \mathbb{Q}_{>0}$ ), where  $f_1, \dots, f_k$  are local generators of  $\mathcal{I}$  and  $v$  is a smooth function. (Refer to [Dem] for details.)

**2.2. Pull-back of  $(1, 1)$ -current.** We can handle  $(1, 1)$ -currents easier than high degree currents. For example we can define the pull-back of a  $d$ -closed  $(1, 1)$ -current. Let  $f : Z \rightarrow M$  be a holomorphic map between complex manifolds. Assume that the image of  $Z$  by  $f$  is not contained in the polar set of  $T$ . Let  $T = \theta + dd^c\varphi$  be a  $d$ -closed  $(1, 1)$ -current in the class  $\alpha \in H^{1,1}(M, \mathbb{R})$ . Here  $\theta$  is a smooth  $(1, 1)$ -form in the class  $\alpha$  and  $\varphi$  is a  $L^1$ -function on  $M$ . Then the pullback of  $T$  is defined to be  $f^*T := f^*\theta + dd^cf^*\varphi$  by using the potential  $\varphi$  of  $T$ . In particular, we can restrict a  $d$ -closed  $(1, 1)$ -current to any submanifold which is not contained in the polar set.

**2.3. Multiplier ideal sheaves and Skoda's lemma.** In this paper, multiplier ideal sheaves is used. We denote by  $\mathcal{J}(\|kD\|)$  the asymptotic multiplier ideal sheaves associated to a divisor  $kD$  and by  $\mathcal{I}(kT)$  the multiplier ideal sheaves associated to a  $d$ -closed current  $kT$ . We can refer to [DEL00], [Dem] for more details. The Skoda's Lemma implies that a multiplier ideal sheaf is estimated by the Lelong numbers (see [Dem, Lemme 6.6]).

**Lemma 2.1.** [Skoda's Lemma] (a) If  $\nu(T, x) < 1$ , then  $e^{-2\varphi}$  is integrable in a neighborhood of  $x$ . In particular  $\mathcal{I}(T)_{M,x} = \mathcal{O}_{M,x}$ .  
 (b) If  $\nu(T, x) > n + s$  for some positive integer  $s$ , then  $e^{-2\varphi} \geq C|z - x|^{-2n-2s}$  in a neighborhood of  $x$ . In particular  $\mathcal{I}(T)_{M,x} \subseteq \mathcal{M}_{M,x}^{s+1}$ , where  $\mathcal{M}_{M,x}$  is the maximal ideal of  $\mathcal{O}_{M,x}$ .

**2.4. Lebesgue decompositions.** We can always locally see a positive current  $T$  as the differential form with positive measure coefficients. A positive measure admits a Lebesgue decomposition into the absolutely continuous part and the singular part with respect to the Lebesgue measure. This decomposition is local. However we can globally glue the absolutely continuous part and the singular part by the uniqueness of a Lebesgue decomposition. So we obtain the decomposition  $T = T_{\text{ac}} + T_{\text{sing}}$ , where  $T_{\text{ac}}$  is the form with  $L^1_{\text{loc}}$ -function coefficients. Then we have  $T_{\text{ac}} \geq \gamma$  when  $T \geq \gamma$  for some smooth form  $\gamma$ . In particular the absolutely continuous part  $T_{\text{ac}}$  is positive, when  $T$  is a positive current. Note that  $T_{\text{ac}}$  is not  $d$ -closed in general even if  $T$  is  $d$ -closed. Then we can define the wedge  $T_{\text{ac}}^k$  almost everywhere since  $T_{\text{ac}}$  is a positive form with  $L^1_{\text{loc}}$ -function coefficients (Refer to [Bou02] for more details).

**2.5. Approximations of currents.** Fix a Kähler form  $\omega$  on  $M$ . Let  $T = \theta + dd^c\varphi$  be a  $(1,1)$ -current in the class  $\alpha \in H^{1,1}(M, \mathbb{R})$ , where  $\theta$  is a smooth  $(1,1)$ -form in the class  $\alpha$ . We assume that  $T \geq \gamma$  holds for a smooth form  $\gamma$ . Then we can approximate  $T$  by smooth forms in the following sense:

**Theorem 2.2.** [Dem82, THÉORÈME9.1] *There exist a decreasing sequence of smooth functions  $\varphi_k$  converging to  $\varphi$  such that if we set  $T_k = \theta + dd^c\varphi_k \in \alpha$ , we have*

- (a)  $T_k \rightarrow T$  weakly and  $T_k \rightarrow T_{\text{ac}}$  almost everywhere .
- (b)  $T_k \geq \gamma - C\lambda_k\omega$ , where  $C$  is a positive constant depending on  $(M, \omega)$  only, and  $\lambda_k$  is a decreasing sequences of continuous functions such that  $\lambda_k \searrow \nu(T, x)$  for all  $x \in M$ .

Roughly speaking Theorem 2.2 says that it is possible to smooth a given current  $T$  insides the class  $\alpha$ , but only with the loss of positivity controled by the Lelong numbers of  $T$ . By the proof of Theorem 2.2 in [Dem82], we can add the following property (d).

- (c) *When  $T$  is smooth on some neighborhood of  $x_0 \in M$ ,  $T_k \rightarrow T_{\text{ac}}$  is a pointwise convergence on some neighborhood of  $x_0$ .*

The following theorem says that it is possible to approximate a given current  $T$  with currents with analytic singularities without the loss of positivity.

**Theorem 2.3.** [Dem92], [Bou02, Theorem 2.4] *There exists a decreasing sequence of functions  $\varphi_k$  with analytic singularities converging to  $\varphi$  such that if we set  $T_k = \theta + dd^c\varphi_k \in \alpha$ , we have*

- (a')  $T_k \rightarrow T$  weakly and  $T_{k,\text{ac}} \rightarrow T_{\text{ac}}$  almost everywhere.
- (b')  $T_k \geq \gamma - \varepsilon_k\omega$ , where  $\varepsilon_k$  is a positive number converging to zero.
- (c') *The Lelong number  $\nu(T_k, x)$  increase to  $\nu(T, x)$  uniformly with respect to  $x \in M$ .*

In the proof of [Bou02, Theorem 2.4], the convergence  $T_{k,\text{ac}} \rightarrow T_{\text{ac}}$  in (a') is proved only from the property (a) in Theorem 2.2. Therefore we can add the following property (d') from the property (c).

(d') When  $T$  is smooth on some neighborhood of  $x_0 \in M$ ,  $T_k \rightarrow T_{\text{ac}}$  is pointwise convergence on some neighborhood of  $x_0$ .

This consideration shows the following corollary.

**Corollary 2.4.** *Assume that  $T$  has analytic singularities whose polar set does not contain a closed analytic set  $V$ . Then  $T_k$  in Theorem 2.3 satisfies the following:*

$$(T_k|_{V_{\text{reg}}})_{\text{ac}} \rightarrow (T|_{V_{\text{reg}}})_{\text{ac}} \quad \text{for almost point in } V_{\text{reg}}.$$

**2.6. Restricted volumes and properties.** For simplicity, we denote by  $H^0(X|V, \mathcal{O}_X(D))$

$$H^0(X|V, \mathcal{O}_X(D)) := \text{Im} \left( H^0(X, \mathcal{O}_X(D)) \rightarrow H^0(V, \mathcal{O}_V(D)) \right).$$

The restricted volume of  $D$  along  $V$  is defined to be

$$\text{vol}_{X|V}(D) = \limsup_{k \rightarrow \infty} \frac{\dim H^0(X|V, \mathcal{O}_X(D))}{k^d/d!}.$$

Note that when  $V$  is the ambient space  $X$ , the restricted volume of  $D$  is the usual volume. When  $V$  is not contained in  $\mathbb{B}_+(D)$ , many results for usual volumes are extended to restricted volumes. In this paper, we refer the basic properties of restricted volumes to [ELMNP]. Here  $\mathbb{B}_+(D)$  (*resp.*  $\mathbb{B}_-(D)$ ) means the augmented base locus (*resp.* the restricted base locus) of  $D$  (see [ELMNP2], [ELMNP3]). The non-Kähler locus  $E_{nK}(\alpha)$  (*resp.* non-nef locus  $E_{nn}(\alpha)$ ) of a class  $\alpha$  in  $H^{1,1}(X, \mathbb{R})$  is the analytic counter part of the augmented base locus (*resp.* restricted base locus) (see [Bou04]). When  $\alpha$  is the first Chern class  $c_1(D)$  of some divisor  $D$ , the augmented base locus (*resp.* restricted base locus) coincide with the non-Kähler locus (*resp.* the non-nef locus). If  $\alpha$  is a big class,  $E_{nn}(\alpha)$  coincide with  $E_+(T_{\min}) := \{x \in X \mid \nu(T_{\min}, x) > 0\}$ , where  $T_{\min}$  is a minimal singular current in  $\alpha$  (see [Bou04, proposition 3.8]).

**2.7. Divisorial Zariski decomposition.** In this subsection, we recall the definition of divisorial Zariski decompositions. Divisorial Zariski decomposition for a big divisor coincides with  $\sigma$ -decompositions. Divisorial Zariski decompositions are studied in [Bou04] and  $\sigma$ -decompositions are studied in [Nak].

Let  $\alpha$  be a pseudo-effective class in  $H^{1,1}(M, \mathbb{R})$ . Then an effective  $\mathbb{R}$ -divisor  $N$  is defined as follows:

$$N := \sum_{F: \text{prime div}} \nu(T_{\min}, F) F,$$

where  $T_{\min}$  is a minimal singular current in  $\alpha$ . Here  $\nu(T_{\min}, F)$  is the Lelong number along  $F$ , which is defined by  $\inf_{x \in F} \nu(T_{\min}, x)$ . The class  $P$  is defined by  $P := \alpha - \{N\}$ . Here  $\{N\}$  denotes the class of  $N$ . Then the decomposition  $\alpha = P + \{N\}$  is said to be the divisorial Zariski decomposition of  $\alpha$ .  $P$  is called the positive part and  $\{N\}$  is called the negative part of  $\alpha$ . In general the positive part  $P$  is nef in codimension 1. (that is, the codimension of the non-nef locus is greater than 1.) We say that  $\alpha$  admits a Zariski decomposition if the positive part  $P$  is nef. If  $\alpha$  is the class of some divisor, this definition coincides with that of the divisor (see section

1). For example if  $M$  is surface, any big class admits a Zariski decomposition (see [Bou04, section 4]). By the construction of  $N$ , positive currents in  $\alpha$  and positive currents in  $P$  are identified by the following correspondence:

$$\alpha \ni T \longmapsto T - [N] \in P.$$

### 3. RESTRICTED VOLUMES AND ZARISKI DECOMPOSITIONS

**3.1. Positive parts and restricted volumes.** The main purpose in this section is to prove Theorem 1.1. In this section, we assume  $D$  is big. We begin to prepare for the proof of Theorem 1.1. Consider the divisorial Zariski decomposition  $D = P + N$  of  $D$ . Since  $D$  is big, the divisorial Zariski decomposition coincide with the  $\sigma$ -decomposition. Note that  $\mathbb{B}_+(D) = \mathbb{B}_+(P)$  and  $\text{Supp}(N) \subseteq \mathbb{B}_+(D)$ . The following proposition shows that the restricted volume of  $D$  can be computed by the positive part  $P$ .

**Proposition 3.1.** *Let  $W \subseteq X$  be an irreducible subvariety which is not contained in  $\mathbb{B}_+(D)$ . Then the equality  $\text{vol}_{X|W}(D) = \text{vol}_{X|W}(P)$  holds.*

*Remark 3.2.* In general  $P$  is an  $\mathbb{R}$ -divisor. Then  $\text{vol}_{X|W}(P)$  is defined by the limit of the restricted volumes of  $\mathbb{Q}$ -divisors which converges to  $P$  in Néron-Severi group. By the continuity of restricted volumes ([ELMNP, Theorem 5.2]),  $\text{vol}_{X|W}(P)$  does not depend on the choice of  $\mathbb{Q}$ -divisors which converges to  $P$ .

*Proof.* By the bigness of  $D$ , we can take an effective  $\mathbb{Q}$ -divisor  $D'$  with  $D \sim_{\mathbb{Q}} D'$ . By multiplying positive integer, we can assume that  $D$  is effective divisor. Since  $W$  is not contained in  $\mathbb{B}_+(D)$ , we can assume  $W \not\subseteq \text{Supp}(D)$ . Moreover  $W$  are not contained in  $\text{Supp}(N \cup P)$ , since  $\text{Supp}(N \cup P)$  are contained in  $\text{Supp}(D)$ . From [Bou04, Theorem 5.5], there exists a canonical isomorphism  $H^0(X, \mathcal{O}_X(\lfloor kP \rfloor)) \cong H^0(X, \mathcal{O}_X(kD))$  by multiplying the section  $e_k$  for a positive integer  $k$ , where  $e_k$  is the standard section of the effective divisor  $\lfloor kN \rfloor$ . We consider the following commutative diagram:

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X(\lfloor kP \rfloor)) & \xrightarrow{\cdot e_k} & H^0(X, \mathcal{O}_X(kD)) \\ f \downarrow & & g \downarrow \\ H^0(W, \mathcal{O}_W(\lfloor kP \rfloor)) & \xrightarrow{\cdot e_k|_W} & H^0(W, \mathcal{O}_W(kD)) \end{array}$$

$f$  and  $g$  are the restriction maps. This diagram induces the map  $\text{Im}(f) \longrightarrow \text{Im}(g)$ . This map is surjective since the vertical map is an isomorphism. Moreover the map is injective since  $e_k|_W$  is a nonzero section from  $W \not\subseteq \text{Supp}(N)$ . So we have

$$\text{vol}_{X|W}(D) = \limsup_{k \rightarrow \infty} \frac{h^0(X|W, \mathcal{O}_X(\lfloor kP \rfloor))}{k^d/d!}.$$

When  $P$  is a  $\mathbb{Q}$ -divisor, Proposition 3.1 follows from this equation and the homogeneity of restricted volumes ([ELMNP, Lemma 2.2]). When  $P$  is an  $\mathbb{R}$ -divisor, we can reduce to a  $\mathbb{Q}$ -divisor by the continuity of restricted volumes. (Note that  $\lfloor kP \rfloor/k \rightarrow P$  ( $k \rightarrow \infty$ ).)  $\square$

**Corollary 3.3.** *Let  $W \subseteq X$  be an irreducible subvariety of dimension  $d$  which is not contained in  $\mathbb{B}_+(D)$ . When  $D$  admits a Zariski decomposition (that is, the positive part  $P$  is nef), the equality  $\text{vol}_{X|W}(D) = (W \cdot P^d)$  holds.*

*Proof.* By Theorem 3.1, we obtain  $\text{vol}_{X|W}(D) = \text{vol}_{X|W}(P)$ . Since  $P$  is nef, there exists ample  $\mathbb{Q}$ -divisors  $B_k$  such that  $B_k \rightarrow P$  in the Néron-Severi group. The Ampleness of  $B_k$  give  $\text{vol}_{X|W}(B_k) = (W \cdot B_k^d)$ . Therefore the continuity of restricted volumes shows

$$\begin{aligned} \text{vol}_{X|W}(D) &= \text{vol}_{X|W}(P) \\ &= \lim_{k \rightarrow \infty} \text{vol}_{X|W}(B_k) \\ &= \lim_{k \rightarrow \infty} (W \cdot B_k^d) = (W \cdot P^d). \end{aligned}$$

□

**3.2. Proof of the main Theorem.** In this subsection we complete the proof of Theorem 1.1. We take subvarieties  $V$  and  $V'$  on  $X$  such that  $V \sim V'$  and  $V, V' \not\subseteq \mathbb{B}_+(D)$ . Assume that  $D$  admits a Zariski decomposition. By Corollary 3.3 the restricted volume of  $D$  can be computed by the intersection number of the positive part. That is,  $\text{vol}_{X|V}(D) = (V \cdot P^d)$  and  $\text{vol}_{X|V'}(D) = (V' \cdot P^d)$  holds. So  $V \sim V'$  shows  $(V \cdot D^d) = (V' \cdot D^d)$ . Hence  $\text{vol}_{X|V}(D) = \text{vol}_{X|V'}(D)$  holds if  $D$  admits a Zariski decomposition.

Next we prove the converse direction. Assume the condition (2). Moreover we assume that  $P$  is not nef. Then the restricted base locus  $\mathbb{B}_-(P)$  is not empty. Fix a very ample divisor  $A$  on  $X$  and an arbitrary point  $x_0$  in  $\mathbb{B}_-(P)$ . Then we can take irreducible smooth curves  $C$  and  $C'$  with the following properties:

- (1)  $C$  and  $C'$  are not contained in the augmented base locus  $\mathbb{B}_+(D)$ .
- (2)  $C$  intersects with the restricted base locus  $\mathbb{B}_-(P)$  at  $x_0 \in X$ .
- (3)  $C'$  does not intersect with the restricted base locus  $\mathbb{B}_-(P)$ .
- (4)  $C$  and  $C'$  are the complete intersections of the linear system of  $A$ .

We can take such curves. Since the codimension of the restricted base locus  $\mathbb{B}_-(P)$  is greater than one, a very general complete intersection curve  $C'$  satisfies the properties (1), (3) by Bertini's Theorem. (We can not take  $C'$  which does not intersect  $\mathbb{B}_-(D)$  in general.) [Zha09, Theorem 2.5] assure that a general hyperplane passing through  $x_0$  is irreducible and smooth. Therefore we can take a smooth curve  $C$  which satisfies the properties (1), (2).

Now  $C$  and  $C'$  are numerically equivalent since  $C$  and  $C'$  are the complete intersections of the same linear system. So the equality  $\text{vol}_{X|C}(P) = \text{vol}_{X|C'}(P)$  holds by the assumption and Theorem 3.1. Therefore it is sufficient to prove the following claim for the contradiction to the assumption that  $\mathbb{B}_-(P)$  is not empty.

**Lemma 3.4.** (A)  $\text{vol}_{X|C'}(P) = (C' \cdot P)$  ( $= (C \cdot P)$ ).  
 (B)  $\text{vol}_{X|C}(P) < (C \cdot P)$ .



*Proof.* For simplicity, we assume  $P$  is a  $\mathbb{Q}$ -divisor. When  $P$  is an  $\mathbb{R}$ -divisor, the similar argument give the same results by using continuity of restricted volumes. Fix a positive integer  $a$  such that  $aP$  is a  $\mathbb{Z}$ -divisor.

(A): Then from [ELMNP, Theorem 2.13], the restricted volume of  $P$  can be computed as follows:

$$\mathrm{vol}_{X|C'}(P) = \limsup_{\ell \rightarrow \infty} \frac{h^0(C', \mathcal{O}_{C'}(\ell aP) \otimes \mathcal{J}(\|\ell aP\|)|_{C'})}{\ell a},$$

Since  $P$  is a big divisor, the restricted base locus of  $P$  equals to the set  $\{x \in X \mid \nu(T_{\min}, x) > 0\}$ , where  $T_{\min}$  is a minimal singular current in  $c_1(P)$ . So the Lelong number of every point  $x$  on  $C'$  is zero by the property (3). This implies  $\mathcal{I}_+(\ell T_{\min})$  and  $\mathcal{I}(\ell T_{\min})$  are trivial along  $C'$  by Skoda's Lemma. This shows  $\mathcal{J}(\|\ell P\|)$  is trivial along  $C'$  by [DEL00, Theorem 1.1]. Hence we obtain

$$\mathrm{vol}_{X|C'}(P) = \limsup_{\ell \rightarrow \infty} \frac{h^0(C', \mathcal{O}_{C'}(\ell aP))}{\ell a}.$$

Since  $C'$  is not contained in  $\mathbb{B}(P)$ ,  $P$  is an ample divisor on  $C'$ . By Riemann-Roch formula on compact Riemann surfaces, we have  $\mathrm{vol}_{X|C'}(P + A_k) = ((P + A_k) \cdot C')$ . In fact, Riemann-Roch formula shows

$$h^0(C', \mathcal{O}_{C'}(\ell aP)) = h^1(C', \mathcal{O}_{C'}(\ell aP)) + (\ell aP \cdot C') - g + 1,$$

where  $g$  is a genus of  $C'$ . By the ampleness of  $P$ , the first cohomology vanishes for sufficiently large  $\ell$ . So we obtain  $\mathrm{vol}_{X|C'}(P) = (P \cdot C')$ . Therefore (A) in Lemma 3.4 holds.

(B): By the same argument of (A), we get the following equality:

$$\mathrm{vol}_{X|C}(P) = \limsup_{\ell \rightarrow \infty} \frac{h^0(C, \mathcal{O}_C(\ell aP) \otimes \mathcal{J}(\|\ell aP\|)|_C)}{\ell a},$$

Now we estimate the asymptotic multiplier ideal by using Skoda's Lemma. First of all, we obtain  $\mathcal{J}(\|kP\|) \subseteq \mathcal{I}(kT_{\min})$  for every positive integer  $k$  from the minimal singularity of  $T_{\min}$ , where  $T_{\min}$  is a minimal singular current in  $c_1(P)$ . Since  $\nu(T_{\min}, x_0)$  is positive by the property (2), we can take a positive rational number  $p/q$  which is smaller than  $\nu(T_{\min}, x_0)$ . So we obtain  $p\ell a < \nu(q\ell aT_{\min}, x_0)$ . Skoda's Lemma implies

$\nu(kqT_{\min})_{x_0} \subseteq \mathcal{M}_{x_0, X}^{p\ell a - n + 1}$ , where  $\mathcal{M}_{x_0, X}$  is the maximal ideal in  $\mathcal{O}_{x_0, X}$ . So we obtain

$$\begin{aligned} \text{vol}_{X|C}(P) &\leq \limsup_{k \rightarrow \infty} \frac{h^0(C, \mathcal{O}_C(\ell a P) \otimes \mathcal{M}_{x_0, X}^{p\ell a - n + 1}|_C)}{\ell a} \\ &\leq \limsup_{k \rightarrow \infty} \frac{h^0(C, \mathcal{O}_C(\ell a P) \otimes \mathcal{M}_{x_0, C}^{p\ell a - n + 1})}{\ell a} \\ &= \limsup_{k \rightarrow \infty} \frac{h^0(C, \mathcal{O}_C(\lfloor \ell a P \rfloor - (p\ell a - n + 1)[x_0]))}{\ell a}, \end{aligned}$$

where  $[x_0]$  is a divisor on  $C$  defined by  $x_0$ . By the same argument of (A), we obtain  $\text{vol}_{X|C}(P) < (C \cdot P) - p/q$ .  $\square$

By the proof of Lemma 3.5 (B), the following corollary is proved.

**Corollary 3.5.** *Let  $C$  be an irreducible smooth curve. Assume that  $C$  is not contained in the augmented base locus of  $D$ . Then the following inequality holds.*

$$\text{vol}_{X|C}(D) \leq (C \cdot D) - \sum_{x \in C \cap \mathbb{B}_-(D)} \nu(T_{\min}, x)$$

#### 4. EXPRESSION OF RESTRICTED VOLUMES BY CURRENT INTEGRATION

**4.1. Proof of the expression formula.** The main purpose of this subsection is to prove Theorem 1.2. Before the proof of Theorem 1.2, we must show that the integral  $\int_{V_{\text{reg}}} (T|_{V_{\text{reg}}})_{\text{ac}}^d$  is always finite.

**Proposition 4.1.** *Let  $W$  be an irreducible subvariety of dimension  $d$  on a compact Kähler manifold  $Y$  and  $S$  a  $d$ -closed positive  $(1, 1)$ -current on  $Y$ . Assume that the polar set of  $S$  does not contain  $W$ . Then the integral  $\int_{W_{\text{reg}}} (S|_{W_{\text{reg}}})_{\text{ac}}^d$  is finite.*

*Proof.* Note the restriction  $S|_{W_{\text{reg}}}$  is  $d$ -closed positive. Hence the integral  $\int_W (S|_W)_{\text{ac}}^d$  is finite by [Bou02, Lemma 2.11] when  $W$  is non-singular. Therefore it is enough to show the integral is finite when  $W$  has singularities. Let  $\mu : \widetilde{W} \subseteq \widetilde{X} \rightarrow W \subseteq X$  be the embedded resolution of  $W \subseteq X$ . That is,  $\mu : \widetilde{X} \rightarrow X$  is a modification and the restriction  $\mu : \widetilde{W} \rightarrow W$  is the resolution of singularities of  $W$ . Then the following lemma assures Proposition 4.1 holds. In fact,  $\int_{\widetilde{W}} ((\mu^* S)|_{\widetilde{W}})_{\text{ac}}^d$  is finite, since  $\widetilde{W}$  is non-singular.  $\square$

**Lemma 4.2.**

$$\int_{W_{\text{reg}}} (S|_{W_{\text{reg}}})_{\text{ac}}^d = \int_{\widetilde{W}} ((\mu^* S)|_{\widetilde{W}})_{\text{ac}}^d$$

*Proof.* The map  $\widetilde{W} \xrightarrow{\mu} W$  is an isomorphism except some subvarieties. Therefore  $(\mu^* S)|_{\widetilde{W}}$  is identified with  $S|_{W_{\text{reg}}}$  by  $\mu$  on Zariski open set.  $((\mu^* S)|_{\widetilde{W}})_{\text{ac}}$  and  $(S|_{W_{\text{reg}}})_{\text{ac}}$  are absolutely continuous with respect to the Lebesgue measure by the

definition. Hence we obtain  $\int_{W_{\text{reg}}} (S|_{W_{\text{reg}}})_{\text{ac}}^d = \int_{\widetilde{W}} ((\mu^*S)|_{\widetilde{W}})_{\text{ac}}^d$  since a Zariski closed set is measure zero with respect to the Lebesgue measure.  $\square$

*Proof of Theorem 1.2.) (Step1)* First of all we show the inequality  $\geq$  in Theorem 1.2 by using the singular holomorphic Morse inequalities (see [Bon93]) and Proposition 4.4. Proposition 4.4 is proved in the end of this subsection. At first we consider the case when  $V$  is non-singular. For a positive  $(1,1)$ -current  $T$  with analytic singularities in the Chern class  $c_1(D)$  whose singular locus does not contain  $V$ , we obtain the following inequality:

$$\begin{aligned} \text{vol}_{X|V}(D) &= \limsup_{k \rightarrow \infty} \frac{\dim H^0(V, \mathcal{O}_V(kD) \otimes \mathcal{I}(kT_{\min})|_V)}{k^d/d!} \\ &\geq \limsup_{k \rightarrow \infty} \frac{\dim H^0(V, \mathcal{O}_V(kD) \otimes \mathcal{I}(kT)|_V)}{k^d/d!} \\ &\geq \limsup_{k \rightarrow \infty} \frac{\dim H^0(V, \mathcal{O}_V(kD) \otimes \mathcal{I}(kT|_V))}{k^d/d!}. \end{aligned}$$

Here  $T_{\min}$  is a minimal singular current in  $c_1(D)$ . The first equality follows from Proposition 4.4 and the second inequality follows from the restriction formula. By using the singular holomorphic Morse inequality we have

$$\begin{aligned} \text{vol}_{X|V}(D) &\geq \limsup_{k \rightarrow \infty} \frac{\dim H^0(V, \mathcal{O}_V(kD) \otimes \mathcal{I}(kT|_V))}{k^d/d!} \\ &\geq \int_V (T|_V)_{\text{ac}}^d. \end{aligned}$$

Therefore when  $V$  is non-singular, Step 1 is proved.

Next we consider the case when  $V$  has singularities. Then we consider the embedded resolution  $\mu : \widetilde{V} \subseteq \widetilde{X} \rightarrow V \subseteq X$ . Note the augmented base locus of the pull back  $\mu^*D$  does not contain  $\widetilde{V}$  since the restriction  $\mu : \widetilde{V} \rightarrow V$  is isomorphism at a generic point. Then by applying the singular holomorphic Morse inequality and restriction formula to  $\mu^*D$ ,  $\widetilde{V}$  and  $\mu^*T$  again, we obtain

$$\text{vol}_{X|\widetilde{V}}(\mu^*D) \geq \int_{\widetilde{V}} ((\mu^*T)|_{\widetilde{V}})_{\text{ac}}^d.$$

By Lemma 4.1 and [ELMNP, Lemma 6.7] shows the following lemma. By this lemma and the above inequalities, Step 1 is proved even if  $V$  has singularities.

**Lemma 4.3.**

$$\begin{aligned} (1) \quad &\text{vol}_{X|V}(D) = \text{vol}_{X|\widetilde{V}}(\mu^*D) \\ (2) \quad &\int_V (T|_{V_{\text{reg}}})_{\text{ac}}^d = \int_{\widetilde{V}} ((\mu^*T)|_{\widetilde{V}})_{\text{ac}}^d \end{aligned}$$

**(Step2)** We show the converse inequality  $\leq$  to complete the proof of Theorem 1.2. By [ELMNP, Proposition 2.11], for an arbitrary number  $\varepsilon > 0$ , we can find the modification  $\pi_\varepsilon : X_\varepsilon \rightarrow X$  and the expression  $\pi_\varepsilon^*(D) = A_\varepsilon + E_\varepsilon$  such that  $(A_\varepsilon^d \cdot V_\varepsilon) \geq \text{vol}_{X|V}(D) - \varepsilon$ . Here  $A_\varepsilon$  is an ample  $\mathbb{Q}$ -divisor and  $E_\varepsilon$  is an effective  $\mathbb{Q}$ -divisor such that the proper transformation  $V_\varepsilon$  is not contained in  $\text{Supp}(E_\varepsilon)$ . We express the intersection number  $(A_\varepsilon^d \cdot V_\varepsilon)$  by current integration. Let  $\omega_\varepsilon$  be a smooth positive  $(1,1)$ -form on  $X_\varepsilon$  in the first Chern class  $c_1(A_\varepsilon)$ . Since  $\text{Supp}(E_\varepsilon)$  does not contain  $V_\varepsilon$ , we can restrict  $[E_\varepsilon]$  to  $V_\varepsilon$ . Since  $[E_\varepsilon]$  is a positive current, we obtain

$$\begin{aligned} (A_\varepsilon^d \cdot V_\varepsilon) &= \int_{V_\varepsilon} (\omega_\varepsilon|_{V_\varepsilon})^d \\ &\leq \int_{V_\varepsilon} ((\omega_\varepsilon + [E_\varepsilon])|_{V_\varepsilon})_{\text{ac}}^d \\ &= \int_{V_{\text{reg}}} \{(\pi_{\varepsilon*}(\omega_\varepsilon + [E_\varepsilon]))|_{V_{\text{reg}}}\}_{\text{ac}}^d. \end{aligned}$$

The third equality follows from the property that  $\pi_\varepsilon$  is an isomorphism at a general point of  $V$  and the same argument of Lemma 4.2.

Since  $\pi_\varepsilon$  is a modification, the push-forward  $\pi_{\varepsilon*}(\omega_\varepsilon + [E_\varepsilon])$  is the positive current in the Chern class  $c_1(D)$ . However the push-forward may not have analytic singularities. So we want to approximate the push-forward by positive currents with analytic singularities. When we use Theorem 2.3, the positivity of a current can be lost. Note  $(\omega_\varepsilon + [E_\varepsilon])$  is a Kähler current but the push-forward may not be a Kähler current. Hence we can not always take the sequence of approximation by positive currents. Now we consider the approximations of the push-forward  $\pi_{\varepsilon*}(\omega_\varepsilon + [E_\varepsilon])$  after we change the current for the Kähler current.

For simplicity we write  $T_\varepsilon := \pi_{\varepsilon*}(\omega_\varepsilon + [E_\varepsilon])$ . Since  $V$  is not contained in the augmented base locus  $\mathbb{B}_+(D)$ , we can take a Kähler current  $S$  with analytic singularities such that the pole does not contain in  $V$ . Fatou's Lemma assures

$$\begin{aligned} \text{vol}_{X|V}(D) - \varepsilon &\leq (A_\varepsilon^d \cdot V_\varepsilon) \\ &= \int_{V_{\text{reg}}} \liminf_{\delta \rightarrow 0} \{(1 - \delta)(T_\varepsilon|_{V_{\text{reg}}}) + \delta(S|_{V_{\text{reg}}})\}_{\text{ac}}^d \\ &\leq \liminf_{\delta \rightarrow 0} \int_{V_{\text{reg}}} \{(1 - \delta)(T_\varepsilon|_{V_{\text{reg}}}) + \delta(S|_{V_{\text{reg}}})\}_{\text{ac}}^d. \end{aligned}$$

Hence there exists a sufficiently small  $\delta_0 > 0$  with the following:

$$\text{vol}_{X|V}(D) - 2\varepsilon \leq \int_{V_{\text{reg}}} \{(1 - \delta_0)(T_\varepsilon|_{V_{\text{reg}}})_{\text{ac}} + \delta_0(S|_V)_{\text{ac}}\}^d.$$

Note that  $(1 - \delta_0)T_\varepsilon + \delta_0 S$  is a Kähler current in  $c_1(D)$ . By applying the approximation theorem (Theorem 2.3 and Corollary 2.4) to  $(1 - \delta_0)T_\varepsilon + \delta_0 S$ , we can take

positive currents  $\{U_k\}_{k=1}^\infty$  in  $c_1(D)$  with the following properties.

- (1)  $U_k$  has an analytic singularities for every integer  $k$ .
- (2)  $(U_k|_{V_{\text{reg}}})_{\text{ac}} \longrightarrow \{(1 - \delta_0)T_\varepsilon|_{V_{\text{reg}}} + \delta_0 S|_{V_{\text{reg}}}\}_{\text{ac}}$  for almost point in  $V_{\text{reg}}$
- (3)  $U_k$  is a positive current for a sufficiently large  $k$

The property (2) and Fatou's Lemma show

$$\begin{aligned} \text{vol}_{X|V}(D) - 2\varepsilon &\leq \int_{V_{\text{reg}}} \{(1 - \delta_0)(T_\varepsilon|_{V_{\text{reg}}})_{\text{ac}} + \delta_0(S|_{V_{\text{reg}}})_{\text{ac}}\}^d \\ &= \int_{V_{\text{reg}}} \liminf_{k \rightarrow \infty} (U_k|_{V_{\text{reg}}})_{\text{ac}}^d \\ &\leq \liminf_{k \rightarrow \infty} \int_{V_{\text{reg}}} (U_k|_{V_{\text{reg}}})_{\text{ac}}^d. \end{aligned}$$

Therefore we have

$$\text{vol}_{X|V}(D) - 3\varepsilon \leq \int_{V_{\text{reg}}} (U_{k_0}|_{V_{\text{reg}}})_{\text{ac}}^d$$

for a sufficiently large  $k_0$ . Since  $\varepsilon$  is an arbitrary positive number and  $U_{k_0}$  is a positive current with analytic singularities in the Chern class  $c_1(D)$  whose pole of  $U_{k_0}$  does not contain  $V$ . Hence Step 2 is finished. Theorem 1.2 is proved.  $\square$

In the end of this subsection, we prove the following proposition. It is the variation of [ELMNP, Theorem 2.13].

**Proposition 4.4.** *Under the assumption that  $V$  is not contained in  $\mathbb{B}_+(D)$ , the following equality holds.*

$$\text{vol}_{X|V}(D) = \limsup_{k \rightarrow \infty} \frac{h^0(V, \mathcal{O}_V(kD) \otimes \mathcal{I}(kT_{\min})|_V)}{k^d/d!},$$

where  $T_{\min}$  is a minimal singular current in the class  $c_1(D)$ .

*Proof.* [ELMNP, Theorem 2.13] assures the following equality under the assumption.

$$\text{vol}_{X|V}(D) = \limsup_{k \rightarrow \infty} \frac{h^0(V, \mathcal{O}_V(kD) \otimes \mathcal{J}(\|kD\|)|_V)}{k^d/d!}$$

To prove Proposition 4.4, we compare the multiplier ideal sheaf  $\mathcal{I}(kT_{\min})$  with the asymptotic multiplier ideal sheaf  $\mathcal{J}(\|kD\|)$ . By the definition of a minimal singular current, we obtain  $\mathcal{I}(kT_{\min}) \supseteq \mathcal{J}(\|kD\|)$  for all positive integer  $k$ . Hence we obtain the inequality  $\leq$  in Proposition 4.4. To prove the converse inequality we show the following Lemma.

**Lemma 4.5.** *There is an effective divisor  $E$  (independent of  $k$ ) which satisfies the following properties:*

- (i)  $\text{Supp}(E)$  does not contain  $V$ .

(ii)  $\mathcal{I}(kT_{\min}) \cdot \mathcal{O}_X(-E) \subseteq \mathcal{J}(\|kD\|)$  for all  $k$  sufficiently large.

*Proof.* This proof is essentially based on the argument in [DEL00]. Fix a very ample divisor  $A$ . For an arbitrary point  $x \in X$ , there exists a zero dimensional complete intersection  $P_x$  of the linear system  $|A|$  containing  $x$ . The Ohsawa-Takegoshi-Manivel  $L^2$ -extension theorem shows for every divisor  $F$  with a singular hermitian metric  $h$  with the nonnegative curvature  $T_h$ , there exists an ample divisor  $B$  ( $B$  depends only  $A$ ) and the following restriction map is surjective (see [OT87], [Man93]).

$$H^0(X, \mathcal{O}_X(F + B) \otimes \mathcal{I}(T_h)) \longrightarrow H^0(P_x, \mathcal{O}_{P_x}(F + B) \otimes \mathcal{I}(T_h|_{P_x}))$$

Moreover the Ohsawa-Takegoshi-Manivel  $L^2$ -extension claims that for every section on  $P_x$ , the extension is satisfied an  $L^2$ -estimate with a constant independent of  $F$ . Now since the dimension of  $P_x$  is zero, the  $L^2$ -estimate does not depend on  $P_x$ . That is,  $L^2$ -estimate of the extension of a section on  $P_x$  depend only on  $B$ . Since  $B$  depends only  $A$ , the  $L^2$ -estimate of the extension of a section on  $P_x$  depend only on  $A$ .

Since  $D$  is big and  $V$  is not contained in the augmented base locus of  $D$ , we can take  $E \in |k_0D - B|$  with the property (i) by choosing a sufficiently large  $k_0$ . We apply the Ohsawa-Takegoshi-Manivel  $L^2$ -extension theorem to  $F_k := (k - k_0)D + E$  equipped with the singular hermitian metric  $h_{\min}^{\otimes k - k_0} \otimes h_E$ . Here  $h_{\min}$  is a minimal singular hermitian metric and  $h_E$  is a singular hermitian metric defined by the standard section of the effective divisor  $E$ . Then for a sufficiently large  $k$  and a point  $x \in X$ , we obtain the global section  $s_x$  of  $F_k + B \equiv_{\text{lin}} kD$  satisfied the following estimates:

$$\int_X \|s_x\|^2_{h_{\min}^{\otimes k - k_0} \otimes h_E \otimes h_B} \leq C \quad \text{and} \quad |s_x(x)|^2_{h_{\min}^{\otimes k - k_0} \otimes h_E \otimes h_B} = 1,$$

where  $C$  is a constant depending only  $A$  and  $h_B$  is a smooth hermitian metric of  $B$  with the positive curvature. From the second equality, we infer the following equality:

$$|s_x(x)|^2 e^{-2(k - k_0)\varphi_{\min} - 2\varphi_E - 2\varphi_B} = 1,$$

where  $\varphi_{\min}$ ,  $\varphi_E$ ,  $\varphi_B$  is the weight of the hermitian metric  $h_{\min}, h_E, h_B$  respectively. Since  $\varphi_B$  is a smooth function and  $X$  is compact, we obtain

$$\varphi_{\min} + \frac{1}{k - k_0} \varphi_E \leq \frac{1}{k - k_0} \log |s_x(x)| + C.$$

The evaluation map  $H^0(X, \mathcal{O}_X(kD)) \longrightarrow \mathbb{C}$  is a bounded operator on the Hilbert space  $H^0(X, \mathcal{O}_X(kD))$  with  $L^2$ -norm. Moreover the operation norm equals to the Bergman kernel  $\log \sum_j |f_j|^2$ , where  $\{f_j\}$  is the orthonormal basis of  $H^0(X, \mathcal{O}_X(kD))$ .

Therefore we get

$$\frac{1}{k - k_0} \log |s_x(x)| + C \leq \frac{1}{k - k_0} \log \sum_j |f_j| + C$$

This inequality implies that the function  $\frac{1}{k - k_0} \log \sum_j |f_j|$  has less singularities than  $\varphi_{\min} + \frac{1}{k - k_0} \varphi_E$ . Considering the definition of the asymptotic multiplier ideal sheaf, we get the property (ii).  $\square$

We complete the proof of Proposition 4.4 by using Lemma 4.5. From the property (i) in the previous Lemma, we can consider the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_V(kD - E) \longrightarrow \mathcal{O}_V(kD) \longrightarrow \mathcal{O}_{V \cap E}(D) \longrightarrow 0$$

Since the dimension of the intersection  $V \cap E$  is smaller than  $\dim V = d$ , we have

$$\limsup_{k \rightarrow \infty} \frac{\dim H^0(V \cap E, \mathcal{O}_{V \cap E}(kD))}{k^d/d!} = 0.$$

Hence we obtain the following inequality by twisting the restriction to  $V$  of the multiplier ideal sheaf  $\mathcal{I}(kT_{\min})$ .

$$\limsup_{k \rightarrow \infty} \frac{\dim H^0(V, \mathcal{O}_V(kD) \otimes \mathcal{I}(kT_{\min})|_V)}{k^d/d!} \leq \limsup_{k \rightarrow \infty} \frac{\dim H^0(V, \mathcal{O}_V(kD - E) \otimes \mathcal{I}(T_{\min})|_V)}{k^d/d!}$$

Since we chose the effective divisor  $E$  satisfied the property (ii) of the previous Lemma, we obtain

$$\limsup_{k \rightarrow \infty} \frac{\dim H^0(V, \mathcal{O}_V(kD - E) \otimes \mathcal{I}(kT_{\min})|_V)}{k^d/d!} \leq \limsup_{k \rightarrow \infty} \frac{\dim H^0(V, \mathcal{O}_V(kD) \otimes \mathcal{I}(\|kD\|)|_V)}{k^d/d!}.$$

These inequalities assure

$$\mathrm{vol}_{X|V}(D) \geq \limsup_{k \rightarrow \infty} \frac{\dim H^0(V, \mathcal{O}_V(kD) \otimes \mathcal{I}(kT_{\min})|_V)}{k^d/d!}.$$

The proof of Proposition 4.4 is finished.  $\square$

**4.2. Properties of restricted volumes.** Theorem 1.2 implies that the definition of restricted volumes can be extended to a big class on a compact Kähler manifold (see Definition 1.3). Let  $M$  be a compact Kähler manifold,  $V \subseteq M$  a irreducible subvariety of dimension  $d$  and  $\alpha$  a big class in  $H^{1,1}(M, \mathbb{R})$  in this section. In this subsection, we study the properties of the restricted volumes of a class on compact Kähler manifold. .

**Proposition 4.6.** *Assume  $\alpha$  is a nef class and  $V$  is not contained in the non Kähler locus  $E_{nK}(\alpha)$ . Then the restricted volume  $\mathrm{vol}_{M|V}(\alpha)$  is computed by the intersection number  $(\alpha^d \cdot V)$ . That is, the equality  $\mathrm{vol}_{M|V}(\alpha) = (\alpha^d \cdot V)$  holds.*

*Proof.* When  $V$  is non-singular, this proposition is proved by using the similar argument with [Bou02, Theorem 4.1]. The case when  $V$  has singularities is reduced to the non-singular case by same the argument in Lemma 4.2.  $\square$

Proposition 4.7 is generalization of Lemma 3.1 to a class on a compact Kähler manifold. Moreover the proof gives another proof of Lemma 3.1. Let  $\alpha = P + \{N\}$  be the divisorial Zariski decomposition of  $\alpha$ .

**Proposition 4.7.** *Assume that  $V$  is not contained in the non-Kähler locus  $E_{nK}(\alpha)$ . Then  $V$  is not contained in  $E_{nK}(P)$  and the equality  $\text{vol}_{M|V}(\alpha) = \text{vol}_{M|V}(P)$  holds.*

*Proof.* This proposition is based on the following fact. Positive currents in  $\alpha$  and positive currents in  $P$  are identified by the following correspondence.

$$\begin{aligned}\alpha &\longrightarrow P \\ T &\longmapsto T - [N]\end{aligned}$$

Note  $T - [N]$  is a Kähler current if  $T$  is a Kähler current in  $\alpha$ . In fact if  $T \geq \delta\omega$  holds,  $T - \delta\omega - [E]$  is a positive current by considering the Siu decomposition for the positive current  $T - \delta\omega$ . Here  $E$  is an effective divisor defined by  $E := \sum_D \nu(T - \delta\omega, D)D$ . Since  $\omega$  is a smooth form, we get  $\nu(T - \delta\omega, D) = \nu(T, D)$ . Since  $E - N$  is an effective divisor,  $T - [N] \geq \delta\omega$  holds.

First of all we show the following Lemma to prove that  $V$  is not contained in  $E_{nK}(P)$ .

**Lemma 4.8.** *The equality  $E_{nK}(\alpha) = E_{nK}(P)$  holds.*

*Proof.* Note  $\text{Supp}(N)$  is clearly contained in  $E_{nK}(\alpha)$  from the definition. For a point  $x \notin E_{nK}(\alpha)$ , we can take a Kähler current  $T$  in  $\alpha$  with analytic singularities such that  $T$  is smooth at  $x$ . Then  $T - [N]$  is a Kähler current in  $P$ . Since  $x$  is not contained in  $\text{Supp}(N)$ , the Kähler current  $T - [N]$  is smooth at  $x$ . So  $x \notin E_{nK}(P)$  holds.

Conversely for a point  $x \notin E_{nK}(P)$ , we take a Kähler current  $S$  such that  $S$  is smooth on  $x$ . We can assume that  $S \geq \omega$ . We prove that  $x$  is not contained in  $\text{Supp}(N)$ . To prove this, we consider the surjective map:

$$\{\text{smooth real } d\text{-closed } (1, 1)\text{-form}\} \longrightarrow H^{1,1}(M, \mathbb{R}), \quad \theta \mapsto \{\theta\}.$$

We regard the space of smooth real  $d$ -closed  $(1, 1)$ -forms as topological space with Fréchet topology. For a smooth  $(n-1, n-1)$ -form  $\gamma$ , the integral  $\int_M \theta_k \wedge \gamma$  converges to  $\int_M \theta \wedge \gamma$  if  $\theta_k \longrightarrow \theta$  in Fréchet topology. Hence the above map  $\theta \mapsto \{\theta\}$  is continuous by Serre duality. From open mapping theorem the map is an open map.

Since the map is open, for a positive number  $\varepsilon$  we can take a sufficiently small  $\delta > 0$  such that the Chern class  $\delta c_1(N)$  contains a smooth form  $\eta$  with  $-\varepsilon\omega \leq \eta \leq \varepsilon\omega$ . So  $S + \eta + (1 - \delta)[N]$  is a positive current in the class  $\alpha$  and the Lelong number equals to  $\nu((1 - \delta)[N], x)$  since  $S$  is smooth at  $x$ . If  $\nu([N], x)$  is positive, it is contradiction to minimality of  $N$ . So  $x$  is not contained in  $\text{Supp}(N)$ . This implies that the positive current  $S + [N]$  lies in  $\alpha$  and is smooth at  $x$ . Hence  $x$  is not contained in  $E_{nK}(\alpha)$ .  $\square$

By this lemma we can define the restricted volume of  $P$ . Note that  $\text{Supp}([E]|_{V_{\text{reg}}})$  is contained in  $E \cap V$ . This assures Lemma 4.9. The Lemma 4.9 implies the exceptional divisor part  $[N]$  does not effect the integration on  $V$ .



**Lemma 4.9.**  $([E]|_{V_{\text{reg}}})_{\text{ac}} = 0$

Lemma 4.9 shows  $\int_{V_{\text{reg}}} (T|_{V_{\text{reg}}})_{\text{ac}}^d = \int_{V_{\text{reg}}} (T|_{V_{\text{reg}}} - [E]|_{V_{\text{reg}}})_{\text{ac}}^d$ . Therefore Theorem 4.7 is proved by correspondence between positive currents in  $\alpha$  and in  $P$ .  $\square$

The following Proposition says that Fujita's approximation theorem for the restricted volume holds for a class  $\alpha$ . This implies the continuity of a restricted volume.

**Theorem 4.10.** *The restricted volume of the class  $\alpha$  along  $V$  can be approximated by intersection numbers of semi-positive classes. That is, the following equality holds.*

$$\text{vol}_{M|V}(D) = \sup_{\pi^*T=B+[E]} (\{B\}^d \cdot \tilde{V})$$

, where the supremum is taken over all resolution  $\pi : \tilde{M} \rightarrow M$  of the positive current  $T \in \alpha$  with analytic singularities such that  $\pi$  is an isomorphism at a generic point of  $V$  and  $\tilde{V} \not\subseteq \text{Supp}(E)$ . (Here  $\tilde{V}$  denotes the proper transformation of  $V$ .)

*Remark 4.11.* By subtracting a class of exceptional divisors from semi-positive class  $\{B\}$ , we can assume that  $\{B\}$  ranges among Kähler classes.

*Proof.* Let  $T$  be a positive current with analytic singularities in the class  $\alpha$  whose polar set is not contained  $V$ . Then We take the modification  $\mu$  such that  $\mu^*(T) = (B + [E])$ . (We can assume that the map is an isomorphism for generic point on  $V$ .) By Lemma 4.2 and Lemma 4.9 we obtain

$$\begin{aligned} \int_{V_{\text{reg}}} (T|_{V_{\text{reg}}})_{\text{ac}}^d &= \int_{\tilde{V}} (\mu^*T|_{\tilde{V}})_{\text{ac}}^d \\ &= \int_{\tilde{V}} ((B + [E])|_{\tilde{V}})_{\text{ac}}^d \\ &= \int_{\tilde{V}} B_{\text{ac}}^d = (\{B\}^d \cdot \tilde{V}). \end{aligned}$$

Therefore we get  $\text{vol}_{M|V}(\alpha) = \sup(\{B\}^d \cdot \tilde{V})$  from the definition of a restricted volume of the class  $\alpha$  along  $V$ .  $\square$

To show the continuity of a restricted volume, we consider the domain of a restricted volume for a subvariety  $V$ . Moreover we prove the convexity of the domain and concavity of restricted volumes.

**Definition 4.12.** For a subvariety  $V$ , the domain of a restricted volume is defined to be  $\text{Big}^V(M) := \{\beta \in H^{1,1}(M, \mathbb{R}) \mid V \not\subseteq E_{nK}(\beta)\}$ .

**Proposition 4.13.** (1)  $\text{Big}^V(M)$  is a open convex domain in  $H^{1,1}(M, \mathbb{R})$   
 (2) For  $\beta_1, \beta_2 \in \text{Big}^V(M)$ , the following inequality holds.

$$\text{vol}_{M|V}(\beta_1 + \beta_2)^{1/d} \geq \text{vol}_{M|V}(\beta_1)^{1/d} + \text{vol}_{M|V}(\beta_2)^{1/d}$$

*Proof.* (1): The concavity is directly followed from  $E_{nK}(\beta + \beta') \subseteq E_{nK}(\beta') \cup E_{nK}(\beta)$  and  $E_{nK}(\beta) = E_{nK}(k\beta)$  for  $k \geq 0$ . For given a class  $\beta$ , we can prove  $E_{nK}(\beta') \subseteq E_{nK}(\beta)$  for every class  $\beta'$  in a suitable open neighborhood of  $\beta$  by using the argument in Lemma 4.8. This assures the domain is open.

(2) Boucksom shows the log concavity for the volume of a transcendental class in [Bou02]. Hence Proposition 4.6 shows a restricted volume has log concavity for nef classes. We can reduce to the case of nef classes by using the Proposition 4.10.  $\square$

**Corollary 4.14.** *The following map is continuous.*

$$\begin{aligned} \text{vol}_{M|V} : \text{Big}^V(M) &\longrightarrow \mathbb{R} \\ \beta &\longmapsto \text{vol}_{M|V}(\beta) \end{aligned}$$

*Proof.* It is well-known fact that a concave function on an open convex set  $\mathbb{R}^N$  is continuous. Therefore Corollary followed from Proposition 4.13.  $\square$

**4.3. Another proof of Theorem 1.1.** In this subsection we prove Theorem 1.4 by using the expression of a restricted volume with current integration. This proof gives another proof of Theorem 1.1. Let  $\alpha \in H^{1,1}(X, \mathbb{R})$  be a big class on  $X$  and  $\alpha = P + \{N\}$  be the divisorial Zariski decomposition. Note  $E_{nK}(\alpha) = E_{nK}(P)$  holds by Lemma 4.8. Hence we can consider the restricted volume of  $P$  along  $V$ .

*Proof of Theorem 1.4.)* The principle of the proof is essentially one of Theorem 1.1. By Proposition 4.7 we get  $\text{vol}_{X|V}(\alpha) = \text{vol}_{X|V}(P)$  for an irreducible subvariety  $V$  with  $V \not\subseteq E_{nK}(\alpha)$ . Moreover Proposition 4.6 claim that  $\text{vol}_{X|V}(P) = (V \cdot P^d)$  holds when  $P$  is nef. Hence the restricted volume are unchangeable for cohomologous subvarieties which is not contained the non-Kähler locus if  $\alpha$  admits a Zariski decomposition.

We prove the converse direction. Assume the non-nef locus  $E_{nn}(P)$  is not empty. Fix a very ample divisor  $A$  on  $X$ . We take smooth curves  $C$  and  $C'$  with the following properties:

- (1)  $C'$  does not intersect with the non-nef locus  $E_{nn}(P)$ .
- (2)  $C$  and  $C'$  are not contained in the non-Kähler locus  $E_{nK}(\alpha)$ .
- (3)  $C$  intersects with the non-nef locus  $E_{nn}(P)$  at  $x_0 \in X$ .
- (4)  $C$  and  $C'$  are the complete intersections of the linear system of  $A$ .

Then we lead the contradictions by proving the following lemma.

**Lemma 4.15.** (A)  $\text{vol}_{X|C'}(\alpha) = (C' \cdot P) (= (C \cdot P))$   
 (B)  $\text{vol}_{X|C}(\alpha) < (C \cdot P)$

Before the proof of Lemma 4.15, we prove Lemma 4.16. Lemma 4.16 assures that the restricted volume is computed by the Lelong number.

**Lemma 4.16.** *Let  $S$  be a positive current with analytic singularities on a smooth curve  $C$ . Then the Siu decomposition coincide with the Lebesgue decomposition of  $S$ . That is,  $S_{ac} = S - \sum_{x \in C} \nu(S, x)[x]$  holds.*

*Proof.* A current  $S$  is locally expressed by  $S = \frac{c}{2} \log(\sum_{i=1}^N |f_i|^2) + \theta$ , where  $\theta$  is a smooth  $(1, 1)$ -form and  $f_i$  is a holomorphic function. We take the maximal effective divisor  $D$  with  $\text{div}(f_i) \geq D \geq 0$  for all  $i = 1, 2, \dots, N$  and the holomorphic function  $g$  with  $\text{div}(g) = D$ . We denote by  $g_i$  a holomorphic function defined by  $g_i = f_i/g$ . Note  $\frac{c}{2} dd^c \log |g|^2 = [cD]$  by Poincarè-Lelong formula. Hence the following decomposition is the Siu decomposition.

$$S = \left\{ \frac{c}{2} \log \left( \sum_{i=1}^N |g_i|^2 \right) + \theta \right\} + \frac{c}{2} \log |g|^2$$

Since  $g_i$  does not have common zero point,  $\log(\sum_{i=1}^N |g_i|^2)$  is smooth. The absolutely continuous part of  $[cD]$  is zero. Hence the decomposition is the Lebesgue decomposition.  $\square$

By using Lemma 4.16, we prove (B) in Lemma 4.15. By the expressions of a restricted volume with current integration and Theorem 4.7 we obtain

$$\text{vol}_{X|C}(\alpha) = \text{vol}_{X|C}(P) = \sup_{T \in P} \int_C (T|_C)_{ac}$$

We take a positive current  $T$  with analytic singularities in the class  $P$  and whose singular locus does not contain  $C$ .  $T|_C$  is a positive current with analytic singularities on  $C$ . By Lemma 4.16 we obtain  $(T|_C)_{ac} = T|_C - \sum_{x \in C} \nu(T|_C, x)[x]$ . Note that the restriction  $T|_C$  lies in  $P|_C$ . This implies that

$$\int_C T|_C = \deg_C(P|_C) = (C \cdot P).$$

In fact  $T|_C$  is expressed  $T|_C = \theta|_C + dd^c \varphi|_C$ , where  $\theta$  is a smooth  $(1, 1)$ -form in  $P$  and  $\varphi$  is a  $L^1$ -function on  $X$ . Hence we obtain

$$\int_C T|_C = (C \cdot P) + \int_C dd^c \varphi|_C.$$

By applying the approximation theorem (Theorem 2.2) to  $\varphi|_C$ , we get smooth functions  $\varphi_k$  on  $C$  such that  $dd^c \varphi_k$  converges to  $dd^c \varphi|_C$  weakly. So  $\int_C dd^c \varphi_k$  converges to  $\int_C dd^c \varphi|_C$ . Here  $\int_C dd^c \varphi_k$  equals to zero by smoothness of  $\varphi_k$  and Stokes's theorem. Hence we have

$$\begin{aligned} \text{vol}_{X|C}(\alpha) &= \sup_{T \in c_1(P)} \left\{ (C \cdot P) - \sum_{x \in C} \nu(T|_C, x) \right\} \\ &= (C \cdot P) - \inf_{T \in P} \sum_{x \in C} \nu(T|_C, x). \end{aligned}$$

The Lelong number of the restriction of a current is more than that of the current. Moreover  $\nu(T_{\min}, x) \leq \nu(T, x)$  holds from the definition of a minimal singular current. Therefore we obtain

$$\mathrm{vol}_{X|C}(\alpha) \leq (C \cdot P) - \sum_{x \in C} \nu(T_{\min}, x).$$

The curve  $C$  intersects with the non-nef locus  $E_{nn}(P)$  at  $x_0$  from the property (3). Hence  $\nu(T_{\min}, x_0)$  is positive. This implies  $\mathrm{vol}_{X|C}(\alpha) \leq (C \cdot P) - \nu(T_{\min}, x_0) < (C \cdot P)$ . Therefore Lemma (B) holds.

We prove (A). By the first half argument we get  $\mathrm{vol}_{X|C'}(\alpha) \leq (C' \cdot P)$ . To show the converse inequality we take a Kähler current  $S \in \alpha$  with analytic singularities. So we can assume  $S \geq \omega$ , where  $\omega$  is a Kähler form on  $X$ . By applying the approximation Theorem (Theorem 2.3) to a minimal singular current  $T_{\min}$  in  $P$ , we get positive currents  $T_k$  with analytic singularities with the following properties.

(b')  $T_k \geq -\varepsilon_k \omega$  and  $\varepsilon_k$  converges to zero.

(c') The Lelong number  $\nu(T_k, x)$  increases to  $\nu(T_{\min}, x)$  for every point  $x \in X$ .

For every positive number  $\delta$ , there is a  $k(\delta)$  such that  $(1 - \delta)T_{k(\delta)} + \delta S$  is a positive current. Since the current has analytic singularities, the inequality

$$\mathrm{vol}_{X|C'}(\alpha) \geq \int_{C'} ((1 - \delta)T_{k(\delta)} + \delta S)|_{C'} \Big|_{\mathrm{ac}}$$

holds for every  $\delta > 0$  by the definition of restricted volumes. The Lelong number of  $T_k$  is zero for every point on  $C$  by the property (3), (b). This shows  $T_k$  is smooth on  $C$ . So we obtain

$$\mathrm{vol}_{X|C'}(\alpha) \geq (1 - \delta)(C' \cdot P) - \delta \int_{C'} (S|_{C'})_{\mathrm{ac}}.$$

holds for every  $\delta$ . When  $\delta$  converges to zero, we get  $\mathrm{vol}_{X|C'}(\alpha) \geq (C' \cdot P)$ .

## 5. ASYMPTOTIC ESTIMATES ON HIGHER COHOMOLOGY

**5.1. Kawamata-Viehweg type vanishing.** The purpose of this section is to prove the asymptotic estimates of higher cohomology of  $\ell D$ . We recall the definition. Let  $k$  be an integer  $k = 0, 1, 2, \dots, n - 1$ .

**Definition 5.1.** A divisor  $D$  is said to be *nef in codimension  $k$*  if the codimension of the restricted base locus  $\mathbb{B}_-(D)$  is greater than  $k$ .

*Remark 5.2.*  $D$  is nef in codimension 0 if and only if the Chern class is a pseudoeffective class.  $D$  is nef in codimension  $n - 1$  if and only if the Chern class is a nef class.

Theorem 1.6 is the accurization of the simplest version of Kawamata-Viehweg vanishing theorem. Lemma 5.3 is usefully worked in the proof of Theorem 1.6 and Corollary 1.8. First of all we prove Lemma 5.3.

**Lemma 5.3.** *Assume that  $D$  is nef in codimension  $k$  and  $k$  is smaller than  $(n - 1)$ . Fix a very ample divisor  $A$  on  $X$ . Then a very general hyperplane  $H \in |A|$  satisfies the following properties.*

- (i)  $H \cap X$  is a smooth projective variety of dimension  $(n - 1)$ .
- (ii) The restriction  $D|_H$  is nef in codimension  $k$  on  $H \cap X$ .

*Proof.* Since  $D$  is nef in codimension  $k$ , for an arbitrary number  $\varepsilon > 0$ , we can take a positive current  $T_\varepsilon$  in the class  $c_1(D)$  with the following properties by the approximation theorem (Theorem 2.3).

- (1)  $T_\varepsilon$  has analytic singularities.
- (2) The codimension of the singular locus of  $T_\varepsilon$  is greater than  $k$ .
- (3)  $T_\varepsilon \geq -\varepsilon\omega$  holds, where  $\omega$  is a Kähler form on  $X$ .

Now we define a countable union of closed analytic sets by

$$Q := \bigcup_{\mathbb{Q} \ni \varepsilon > 0} \bigcup_{\mathbb{Q} \ni c > 0} E_c(T_\varepsilon).$$

Here  $E_c(T_\varepsilon)$  is a Lelong number upper level set, which is defined by  $\{x \in X \mid \nu(T_\varepsilon, x) \geq c\}$ . A theorem of [Siu74] asserts this is a closed analytic set.

**Lemma 5.4.** *For a very general hyperplane  $H \in |A|$ ,  $H$  does not contain analytic set with the codimension  $(k + 1)$  in  $Q$  and  $H \cap X$  is smooth and irreducible.*

*Proof.* Let  $\{C_i\}_{i \in I}$  be a family of analytic sets with codimension  $(k + 1)$  in  $Q$ . For each  $i \in I$ , we define  $Q_i$  by  $Q_i := \{H \in |A| \mid C_i \subseteq H\}$ . Then  $Q_i$  is a (proper) analytic set in  $|A|$  for each  $i$ . Note  $I$  is a countable set. Every hyperplane  $H \in |A| - \bigcup_{i \in I} Q_i$  does not contain an analytic set with the codimension  $(k + 1)$  in  $Q$ . Moreover Bertini's Theorem shows that  $H \cap X$  is smooth and irreducible for a general member  $|A|$ . Lemma 5.4 is concluded.  $\square$

For a very general hyperplane  $H \in |A|$ ,  $T_\varepsilon|_H \in c_1(D|_H)$  and  $T_\varepsilon|_H \geq -\varepsilon\omega|_H$  holds.  $\omega|_H$  is a Kähler form on a smooth projective variety  $X \cap H$ . Hence following lemma concludes Lemma 5.3.

**Lemma 5.5.** *The codimension in  $X \cap H$  of the singular locus of  $T_\varepsilon|_H$  greater than  $k$ .*

Let  $C$  be an analytic set with codimension  $\ell = 0, 1, 2, \dots, k$  on  $X \cap H$ . Then  $C$  is the analytic set with codimension  $\ell + 1$  on  $X$ . When  $\ell + 1$  is smaller than  $k + 1$ ,  $\nu(T_\varepsilon, C) = 0$  for every positive number  $\varepsilon$  by the choice of  $T_\varepsilon$ . When  $\ell + 1$  equals to  $k + 1$ ,  $\nu(T_\varepsilon, C) = 0$  by the choice of  $H$ . Since  $T_\varepsilon$  has analytic singularities, the Lelong number is zero if and only if  $T_\varepsilon$  has smooth at  $x \in X$ . This shows that the singular locus of  $T_\varepsilon|_H$  does not contain an analytic set  $C$  with codimension  $\ell = 0, 1, 2, \dots, k$  on  $X \cap H$ . Hence Lemma 5.3 holds.  $\square$

We prove Theorem 1.6 by using Lemma 5.3. The principle of the proof is to reduce to the ordinary Kawamata-Viehweg vanishing theorem by using Lemma 5.3

and the induction on  $\dim X = n$ .

*Proof of Theorem 1.6.)* When  $n$  is one, bigness of  $D$  implies that  $D$  is ample. In this case, Theorem 1.6 follows from Kodaira vanishing theorem. Hence we can assume that  $n$  is greater than one. If  $k$  is  $(n - 1)$ ,  $D$  is nef. Then Theorem 1.6 follows from the ordinary Kawamata-Viehweg vanishing theorem. Therefore we can assume that  $k$  is smaller than  $(n - 1)$ . It is sufficient to show  $H^p(X, \mathcal{O}_X(-D)) = 0$  ( $0 \leq p \leq k$ ) by Serre duality. We obtain the following short exact sequence for a hyperplane  $H \in |A|$  with the properties in Lemma 5.3.

$$0 \longrightarrow \mathcal{O}_X(-D - A) \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_H(-D) \longrightarrow 0$$

We consider the long exact sequence induced by the above exact sequence. By choosing  $A$  sufficiently positive, we can assume that  $A + D$  is ample on  $X$ . Then Serre duality and Kodaira vanishing theorem show that  $H^\ell(X, \mathcal{O}_X(-D - A)) = 0$ , ( $0 \leq \ell \leq n - 1$ ).  $D|_H$  is nef in codimension  $k$  on  $X \cap H$  by the choice of  $H$ . Therefore  $H^p(H, \mathcal{O}_H(-D)) = 0$  ( $0 \leq p \leq k$ ) holds by the hypothesis of the induction. These show that  $H^p(X, \mathcal{O}_X(-D)) = 0$  ( $0 \leq p \leq k$ ).

□

*Proof of Corollary 1.8.)* Corollary 1.8 is also proved with the induction on  $\dim X = n$ . When  $n$  is one, pseudo-effective line bundle is nef. When the degree of  $D$  is positive, Kodaira vanishing theorem shows that  $H^1(X, \mathcal{O}_X(M + \ell D)) = 0$  for a sufficiently large  $\ell$ . When the degree of  $D$  is zero, Riemann-Roch formula shows  $\dim H^1(X, \mathcal{O}_X(M + \ell D)) = O(1)$ .

We consider when the dimension of  $X$  is greater than one. By choosing a sufficiently positive line bundle  $A$ , we can assume that  $\mathcal{O}_X(M + \ell D - K_X + A)$  is big for all  $\ell$ , since  $D$  is pseudo-effective. Moreover we can take a member  $H$  of  $|A|$  such that  $D|_H$  is nef in codimension  $k$  on  $X \cap H$  by Lemma 5.3. Then we obtain the following short exact sequence.

$$0 \longrightarrow \mathcal{O}_X(M + \ell D) \longrightarrow \mathcal{O}_X(M + \ell D + A) \longrightarrow \mathcal{O}_H(M + \ell D + A) \longrightarrow 0$$

We consider the long exact sequence induced by the above short exact sequence.

Note that  $(E + \ell D - K_X + A)$  is big and nef in codimension  $k$  for all  $\ell > 0$ . So Theorem 1.6 shows  $H^q(X, \mathcal{O}_X(E + \ell D + A)) = 0$  for  $q \geq n - k$ . Therefore we obtain from the long exact sequence

$$\begin{aligned} h^q(X, \mathcal{O}_X(E + \ell D)) &= h^{q-1}(H, \mathcal{O}_H(\mathcal{O}_X(E + \ell D + A))) \quad \text{for } q \geq n - k + 1 \\ \text{and } h^{n-k}(X, \mathcal{O}_X(E + \ell D)) &\leq h^{n-k-1}(H, \mathcal{O}_H(\mathcal{O}_X(E + \ell D + A))). \end{aligned}$$

By hypothesis of induction we have

$$\begin{aligned} h^{q-1}(H, \mathcal{O}_H(\mathcal{O}_X(E + \ell D + A))) &= O(\ell^{(n-1)-(q-1)}) = O(\ell^{n-q}) \quad \text{for } q \geq n - k + 1 \\ \text{and } h^{n-k-1}(H, \mathcal{O}_H(\mathcal{O}_X(E + \ell D + A))) &= O(\ell^{(n-1)-(n-k-1)}) = O(\ell^k) \end{aligned}$$

Therefore Corollary 1.8 holds.

□

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